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Magnitude of magnetic field effects due to a sinusoidal current in a long, thin, straight wire.

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MAGNITUDE OF MAGNETIC FIELD EFFECTS DUE TO A SINUSOIDAL CURRENT IN A LONG, THIN, STRAIGHT WIRE

by

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ABSTRACT

Many electric power lines used for the transmission and distribution of electrical energy are designed to carry large alternating currents. These sinusoidally time-varying currents may be on the order of 1000 amperes. Since these large sinusoidal currents are accompanied by sinusoidal magnetic fields around the wires used to carry them, it is of interest to investigate the magnitude of the effects of these sinusoidal magnetic fields.

Electric power lines may be described as long, thin, straight, current-carrying wires in air. This thesis details the calculation of the magnitude of magnetic field effects due to a sinusoidal current in a long, thin, straight wire in air by using a stationary, rectangular wire loop of thin wire specially oriented with respect to the long, current-carrying wire. The sinusoidal magnetic field surrounding the long, current-carrying wire can induce a sinusoidal electromotive force, and thus a sinusoidal current, in the wire loop. The sinusoidal current so induced in the wire loop, in turn, can induce a sinusoidal magnetic field around the wire in the wire loop. The interaction of the induced sinusoidal magnetic field

around the wire in the wire loop with the sinusoidal magnetic field surrounding the long, current-carrying wire can result in a magnetic force on the wire in the wire loop.

By applying well-known electrical laws, equations are derived in general terms for the magnitude of the sinusoidal magnetic field around the long, current-carrying wire; the magnitude of the sinusoidal voltage induced in the wire loop; the magnitude of the sinusoidal current induced in the wire loop; and the magnitude of the sinusoidal magnetic force acting on the center of mass of the nearest parallel piece of the wire loop. General variables included in the analysis are the frequency, rms value, and phase angle of the sinusoidal current in the long, current-carrying wire; the special orientation of the wire loop with respect to the long, current-carrying wire; and the number of turns in, the length and width of, and the resistance and inductance of the wire in the wire loop. A computer program written and used to calculate the magnitude of the magnetic field effects is described.

Two examples are given based on actual electric power distribution line data to show that induced voltages may be on the order of 150 volts while magnetic forces may be on the order of 0.01 newton.

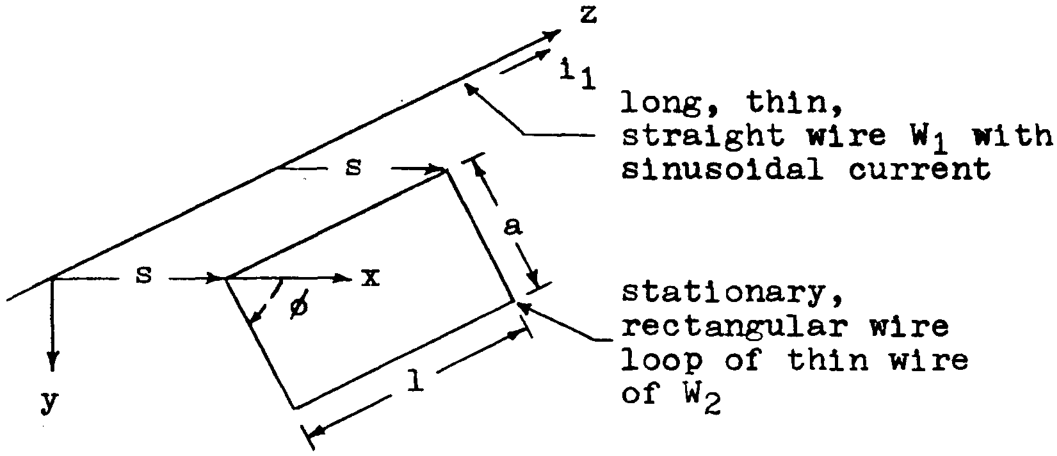
1. INTRODUCTION

A steady-state sinusoidally time-varying (hereafter, simply referred to as "sinusoidal") current in a long, thin, straight wire in air (hereafter, simply referred to as the "long wire") is accompanied by a sinusoidal magnetic field around the long wire.

In order to determine the effects of this sinusoidal magnetic field, consider a stationary, rectangular wire loop of thin wire (hereafter, simply referred to as the "wire loop") specially oriented with respect to the long wire as shown in Figure 1, page 4.

The sinusoidal magnetic field surrounding the long wire can induce a sinusoidal electromotive force (or voltage), and thus a sinusoidal current, in the wire loop. The sinusoidal current so induced in the wire loop, in turn, can induce a sinusoidal magnetic field around the wire in the wire loop. The interaction of the induced sinusoidal magnetic field around the wire in the wire loop with the sinusoidal magnetic field surrounding the long wire can result in a magnetic force on the wire in the wire loop.

Electric power lines used for the transmission and distribution of electrical energy can be approx-



where

W_1 = long, thin, straight wire with wire radius R_1 in meters carrying a sinusoidal current $i_1 = \sqrt{2}\ I_1 \sin(2\pi ft+\alpha)$ with frequency f in hertz, phase angle α in radians, and root-mean-square (rms) current I_1 in amperes

s = distance parallel to the x-axis from the center axis of the long, thin, straight wire to the center axis of the nearest parallel piece of the stationary, rectangular wire loop of thin wire, in meters

ϕ = angle that the plane of the stationary, rectangular wire loop of thin wire makes with the plane containing the long, thin, straight wire and the nearest parallel piece of the stationary, rectangular wire loop of thin wire, in radians between zero and $(\pi/2)$ measured in the $z = 0$ plane from the positive x-axis and counterclockwise as seen from the positive z-axis

W_2 = stationary, rectangular wire loop of thin wire with wire radius R_2 in meters and with N turns, length l in meters, width a in meters, resistance per unit length R_1 in ohms/meter, inductance per unit length L_1 in henries/meter, and impedance per unit length $Z_1 = R_1+j2\pi fL_1$ in ohms/meter.

Figure 1. Special orientation.

imately described as long, thin, straight wires in air carrying a sinusoidal current. As electric power lines may carry large currents on the order of 1000 amperes, it is of interest to investigate the magnitude of the effects of the sinusoidal magnetic field surrounding them.

As a means of investigating the magnitude of these effects of the sinusoidal magnetic field around a long, thin, straight wire carrying a sinusoidal current; a stationary, rectangular wire loop of thin wire as shown in Figure 1, page 4, is used in this paper to detect the magnitude of the induced voltages and currents as well as the magnitude of the magnetic field and magnetic forces due to the sinusoidal current in the long, thin, straight wire.

Not only is a stationary, rectangular wire loop of thin wire a practical and convenient device to use for calculating the magnitude of the effects of the sinusoidal magnetic field due to a sinusoidal current in a long, thin, straight wire, but the wire loop can also be used to represent many physically realizable situations. For example, with appropriate modifications, the wire loop can be used to model: (1) antennas in the vicinity of electric power lines; (2) parallel, isolated wires grounded together at each end near an electric power line; or (3) the human body in the proximity of an electric power line.

By applying well-known electrical laws, equations are derived in general terms for the magnitude of the sinusoidal magnetic field around the long wire; the magnitude of the sinusoidal voltage induced in the wire loop; the magnitude of the sinusoidal current induced in the wire loop; and the magnitude of the sinusoidal magnetic force acting on the center of mass of the nearest parallel piece of the wire loop. General variables included in the analysis are the frequency, rms value, and phase angle of the sinusoidal current in the long wire; the special orientation of the wire loop with respect to the long wire; and the number of turns in the wire loop, the length and width of the wire loop, and the resistance and inductance of the wire in the wire loop.

Finally, a computer program called magforce.ipli is described. The program, based on the equations derived herein, was written and used to calculate the magnitude of the magnetic field effects as a function of given known data. Two examples, calculated by use of the computer program, are given to show the magnitude of magnetic field effects due to a sinusoidal current in a long, thin, straight wire whose current magnitude is typical of that found in electric power distribution lines.

Units of measure for all quantities in this paper are the standard units of the International System of Units (SI).

2. SELECTION OF COORDINATE SYSTEM

To take advantage of the symmetry of Figure 1, page 4, a circular cylindrical coordinate system was chosen and applied to the special orientation of the wire loop with respect to the long wire as shown in Figure 2, page 9.

Variables for Figure 2 are as defined for Figure 1, page 4. In addition, since notation for coordinate systems varies, the notation described in Appendix A has been adopted and used consistently throughout this paper. $\bar{u}_x$, $\bar{u}_y$, and $\bar{u}_z$ are unit vectors in the $\underline{x}$, $\underline{y}$, and $\underline{z}$ directions respectively of a right-handed rectangular cartesian coordinate system and are represented by the ordered vector triplets (1,0,0), (0,1,0), and (0,0,1) respectively. $\bar{u}_r$, $\bar{u}_\theta$, and $\bar{u}_z$ are unit vectors in the $\underline{r}$, $\underline{\theta}$, and $\underline{z}$ directions respectively of a right-handed circular cylindrical coordinate system and are represented by the ordered vector triplets $(\cos\theta, \sin\theta, 0)$, $(-\sin\theta, \cos\theta, 0)$, and $(0,0,1)$ respectively.

The point $P_1:(x,y,z)$ in the right-handed rectangular cartesian coordinate system is equivalent to the point $P_1:(r,\theta,z)$ in the right-handed circular cylindrical coordinate system. $\bar{\rho}$ is the distance

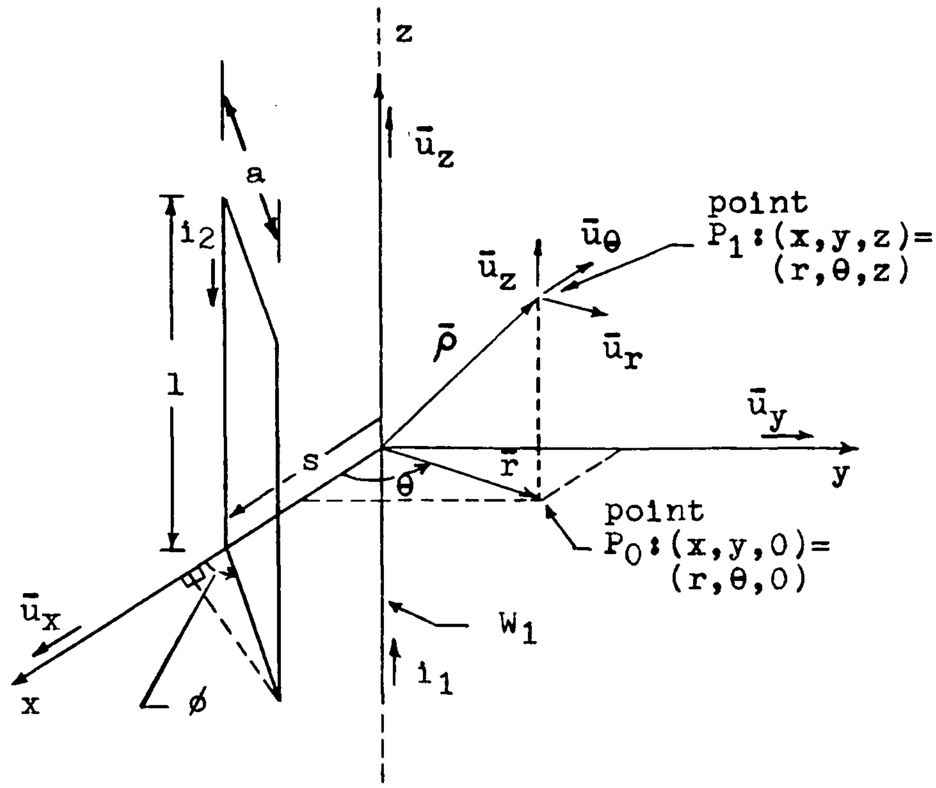


Figure 2. Application of coordinate system.

vector to the point P_1 and is represented by the ordered vector triplet (x,y,z). $\bar{r}$ is the vector projection of $\bar{\rho}$ in the $z = 0$ plane (that is, the distance vector to the point P_0) and is represented by the ordered vector triplet $(x,y,0)$. $\underline{\theta}$ is the angle measured (counterclockwise as seen from the positive $\underline{z}$-axis) from the positive $\underline{x}$-axis to the vector $\bar{r}$. The magnitude of a vector $\bar{v} = (v_1,v_2,v_3)$ is written as $|\bar{v}| = v$.

Finally, the direction of the current $\underline{i}_1$ in the long wire is chosen to be in the positive $\underline{z}$ direction such that at an instant of time when $\underline{i}_1$ is greater than zero, current flow is said to be in the direction of $\bar{u}_z$. Similarly, the direction of the current $\underline{i}_2$ induced in the nearest parallel piece of the wire loop is chosen to be in the negative $\underline{z}$ direction such that at an instant of time when $\underline{i}_2$ is greater than zero, current flow is said to be in the direction of $-\bar{u}_z$.

3. APPLICATION OF AMPERE'S LAW

By Ampere's Law for the magnetic field $\bar{B}$ near a long, straight wire of radius R carrying a current i and surrounded by a medium with relative permeability constant μ_r, the magnetic field lines are concentric circles of radius r around the center axis of the long, straight wire such that

$$\bar{B} = ((\mu_0 \mu_r i)/(2\pi r))\ \bar{u}_\theta \text{ webers/meter}^2 \qquad (1)$$

where

$\bar{B}$ = tangential magnetic field due to current i, in webers/meter^2
μ_0 = permeability constant of a vacuum
= $4\pi \times 10^{-7}$ webers/(ampere meter)
μ_r = relative permeability constant of the surrounding medium
r = radial distance from the center axis of the long, straight wire such that r is greater than R, in meters
i = current in the long, straight wire, in amperes
R = radius of the long, straight wire greater than or equal to zero, in meters.

A restriction on Equation (1) above is that the length of the "long", straight wire must be much greater than the radial distance r from the center axis of the wire.

Refer to Figure 2, page 9. For points at radial distance r from the center axis of the long wire much less than the length of the long wire and for points sufficiently far from the ends of the long wire, there will be no z-dependence. Thus, a typical observation point for the magnetic field around the long wire can be taken in the z = 0 plane of Figure 2.

In particular, for Figure 2, page 9:

$\bar{B} = \bar{B}_1$ = tangential magnetic field due to current $\underline{i}_1$, in webers/meter^2

r = radial distance from the center axis of the long wire such that $\underline{r}$ is greater than $\underline{R}_1$, in meters

$i = i_1 = \sqrt{2}\, I_1 \sin(2\pi f t + \alpha)$, in amperes

$R = R_1$ = radius of the long wire greater than or equal to zero, in meters

and

$$\bar{B}_1 = \sqrt{2}\, \frac{\mu_0 \mu_r I_1}{2\pi r} \sin(2\pi f t + \alpha)\, \bar{u}_\theta \text{ webers/meter}^2. \quad (2)$$

Equation (2) above is the equation for the sinusoidal tangential magnetic field around the long wire of Figure 2, page 9. At an instant of time when the current $\underline{i}_1$ in the long wire is greater than zero and in the positive $\underline{z}$ direction, the magnetic field $\bar{B}_1$ is greater than zero and is in the $+\bar{u}_\theta$ direction.

Equation (2) above is useful in determining the voltage induced in the wire loop by the application of Faraday's Law and the definition of magnetic flux.

4. APPLICATION OF FARADAY'S LAW

By Faraday's Law of Induction, the electromotive force (emf) $\mathcal{E}$ or voltage induced between infinitesimally-opened ends of a wire loop (coil) of N turns with magnetic flux Φ_B passing through each turn of the wire loop (coil) is

$$\mathcal{E} = - d(N\Phi_B)/dt \text{ volts} \tag{3}$$

where

$\mathcal{E}$ = emf induced between infinitesimally-opened ends of a wire loop (coil), in volts
N = number of turns in the wire loop (coil)
Φ_B = magnetic flux passing through each turn of the wire loop (coil) due to magnetic field B, in webers.

In general, N may be a function of time t and Φ_B may be a function of position $\bar{\rho}$ and of time t such that

$$N = N(t)$$
$$\Phi_B = \Phi_B(\bar{\rho}, t)$$

then

$$d(N\Phi_B)/dt = \Phi_B(dN/dt) + N(d\Phi_B/dt)$$
$$d\Phi_B/dt = \partial\Phi_B/\partial t = (\partial\Phi_B/\partial\bar{\rho})(\partial\bar{\rho}/\partial t) + (d\Phi_B/dt)$$

and

$$d(N\Phi_B)/dt = \Phi_B(dN/dt) + N(\partial\Phi_B/\partial\bar{\rho})(\partial\bar{\rho}/\partial t) + N(d\Phi_B/dt).$$

In particular, for Figure 2, page 9, the wire loop is stationary and has a fixed, constant number of turns so that

$\mathcal{E} = e_2$ = emf induced between infinitesimally-opened ends of the wire loop, in volts

N = number of turns in the wire loop

$dN/dt = 0$

$\partial\bar{\rho}/\partial t = 0$

and

$$e_2 = -N(d\Phi_B/dt) \text{ volts.} \tag{4}$$

Before Equation (4) above can be useful in determining the voltage induced in the wire loop, the definition of magnetic flux must be applied to the special orientation of the wire loop with respect to the long wire.

5. APPLICATION OF MAGNETIC FLUX DEFINITION

By the definition of magnetic flux for the magnetic flux Φ_B due to a magnetic field $\bar{B}$ passing through a surface area $\bar{S}$ consisting of elementary infinitesimal surface areas $d\bar{S}$,

$$\Phi_B = \int \bar{B} \cdot d\bar{S} \text{ webers} \tag{5}$$

where

Φ_B = magnetic flux due to magnetic field $\bar{B}$, in webers
$\bar{B}$ = magnetic field, in webers/meter^2
$d\bar{S}$ = infinitesimal surface area, in meter^2.

In Equation (5) above, the surface area $\underline{S}$ under consideration is usually understood; the direction of the infinitesimal surface area is the outward-drawn normal to the surface area; and the dot product $\bar{B} \cdot d\bar{S}$ means the normal component of the magnetic field passing through the infinitesimal surface area $d\bar{S}$ is desired.

In particular, for Figure 2, page 9,

$\Phi_B = \Phi_{B1}$ = magnetic flux passing through the wire loop due to magnetic field B_1, in webers
$\bar{B} = \bar{B}_1$ = tangential magnetic field due to current $\underline{i}_1$, in webers/meter^2.

By Appendix B, in which the surface area $\bar{S}$ under consideration in Equation (5) above is the rectangular area of the wire loop of Figure 2, page 9, and in which the direction of the infinitesimal surface areas is the outward-drawn normal to the rectangular area of the wire loop taken to be in the + $\bar{u}_\phi$ direction,

$$\Phi_{B_1} = \int \bar{B}_1 \cdot d\bar{S} = \sqrt{2}\,\frac{\mu_0 \mu_r l I_1 K_G}{2\pi} \sin(2\pi f t + \alpha) \text{ webers} \quad (6)$$

where

$$K_G = \ln\left|\frac{s+(a\cos\phi)}{s}\right| + \tfrac{1}{2}\ln\left|\frac{s^2+(2sa\cos\phi)+a^2}{s^2+(2sa\cos\phi)+(a^2\cos^2\phi)}\right|$$

ln = natural logarithm to the base e.

Equation (6) above is the equation for the sinusoidal magnetic flux through the wire loop due to the sinusoidal magnetic field $\bar{B}_1$ around the long wire of Figure 2, page 9.

The angle ϕ in Equation (6) above is defined by Appendix B as a constant in the closed interval from zero to ($\pi/2$) inclusive. From the symmetry of Figure 2, page 9, note that angle ϕ could just as easily have been taken in the closed interval from zero to $-(\pi/2)$. This symmetry is contained in the trigonometric identity $\cos(-\phi) = \cos(+\phi)$. However, for simplicity, ϕ in the closed interval zero to ($\pi/2$) has been chosen and used consistently throughout this paper.

For given values of s, a, and ϕ, K_G in Equation (6) above is a geometric constant describing the special orientation of the wire loop with respect to the long wire. Writing the geometric constant $\underline{K_G}$ in terms of (a/s), it is easily shown by considering values of ϕ in the closed interval zero to ($\pi/2$) that $\underline{K_G}$ as a function of ϕ is maximal for a given value of (a/s) if the angle $\phi = 0$. This property of the geometric constant $\underline{K_G}$ confirms the intuitive idea that for a given value of (a/s), the sinusoidal magnetic flux through the wire loop should be maximal when the angle $\phi = 0$.

Substituting Equation (6), page 16, into Equation (4), page 14, and using the trigonometric identity $\cos(2\pi ft+\alpha) = \sin(2\pi ft+\alpha+(\pi/2))$ yields

$$e_2 = \sqrt{2}\ (-N\mu_0\mu_r l I_1 K_G f)\ \sin(2\pi ft+\alpha+(\pi/2)) \text{ volts} \quad (7)$$

where

$$K_G = \ln\left|\frac{s+(a\ \cos\phi)}{s}\right| + \tfrac{1}{2}\ln\left|\frac{s^2+(2sa\ \cos\phi)+a^2}{s^2+(2sa\ \cos\phi)+(a^2\ \cos^2\phi)}\right|.$$

Equation (7) above is the equation for the sinusoidal emf (or voltage) induced in the wire loop of Figure 2, page 9, when the ends of the wire loop are "infinitesimally open-circuited". As such, it is important here to note that this sinusoidal emf induced in the wire loop is taken to be a "source voltage" or "internally-generated voltage". Now if the infinitesimally-opened ends of the wire loop are closed, Lenz's

Law describes the direction of not only the induced emf but also the induced current in the wire loop. As suggested by the negative sign in Equation (7), page 17, for the induced emf in the wire loop resulting from the application of Faraday's Law, Lenz's Law states that the induced emf and the induced current in the wire loop are in such directions as to oppose the change in the sinusoidal magnetic flux through the wire loop.

At an instant of time when the current $\underline{i}_1$ in the long wire is increasingly greater than zero with direction in the positive $\underline{z}$ direction (or equivalently, when the magnetic field $\bar{B}_1$ is increasingly greater than zero in the $+ \bar{u}_\theta$ direction), the induced emf in the wire loop is less than zero but increasing so as to oppose the change in the sinusoidal magnetic flux through the wire loop of Figure 2, page 9.

Equation (7), page 17, is now useful in determining the induced current in the wire loop by the application of Ohm's Law.

6. APPLICATION OF OHM'S LAW

By Ohm's Law, the current $\underline{i}$ through an impedance $\underline{Z}$ with voltage $\underline{v}$ across the impedance is

$$i = \frac{v}{Z} \text{ amperes} \tag{8}$$

where

i = current through the impedance $\underline{Z}$, in amperes
v = voltage across the impedance $\underline{Z}$, in volts
Z = impedance, in ohms.

In particular, for Figure 2, page 9,

$i = i_2$ = current induced through the wire loop, in amperes
$v = e_2$ = emf induced between infinitesimally-opened ends of the wire loop, in volts
$Z = Z_2$ = impedance of the wire loop, in ohms.

By Appendix D, the equation for the sinusoidal current induced in the wire loop of Figure 2, page 9, is

$$i_2 = \sqrt{2}\, I_2 \sin(wt+\alpha+(\pi/2)-(\tan^{-1}(X_1/R_1))) \text{ amperes} \tag{9}$$

where

$$I_2 = \frac{(-N\mu_0\mu_r l I_1 K_G f)}{2(a+1)((R_1)^2+(X_1)^2)^{\frac{1}{2}}} \text{ amperes}$$

$$K_G = \ln\left|\frac{s+(a\cos\phi)}{s}\right| + \tfrac{1}{2}\ln\left|\frac{s^2+(2sa\cos\phi)+a^2}{s^2+(2sa\cos\phi)+(a^2\cos^2\phi)}\right|.$$

$$w = 2\pi f \text{ hertz}$$

$$X_1 = wL_1 \text{ ohms.}$$

As stated previously, the induced current in the wire loop flows in such a direction so as to oppose the change in the sinusoidal magnetic flux through the wire loop (by Lenz's Law), and the direction of the current $\underline{i}_2$ induced in the nearest parallel piece of the wire loop is chosen to be in the negative $\underline{z}$ direction. At an instant of time when $\underline{i}_2$ is greater than zero, current flow is said to be in the direction of $-\bar{u}_z$ of Figure 2, page 9.

Note that of particular importance to the derivation of the induced emf $\underline{e}_2$ of Equation (7), page 17, and the induced current $\underline{i}_2$ of Equation (9), page 19, is the sinusoidal (sinusoidally time-varying) nature of the magnetic field about the long wire. Since both the magnetic field $\bar{B}_1$ and the magnetic flux are sinusoidal, the first time derivative of Φ_{B_1} is nonzero, resulting in both the induced emf $\underline{e}_2$ and the induced current $\underline{i}_2$ in the wire loop being nonzero.

Equation (9), page 19, is useful in determining the magnetic force acting on the wire loop.

7. APPLICATION OF MAGNETIC FORCE EQUATION

By the magnetic force equation, the magnetic force $\bar{F}$ acting on the center of mass of a wire of length $\underline{l}$ carrying a current $\underline{i}$ in an external magnetic field $\bar{B}$ is

$$\bar{F} = i\bar{l} \times \bar{B} \text{ newtons} \tag{10}$$

where

$\bar{F}$ = magnetic force acting on the center of mass of a wire of length $\underline{l}$, in newtons
$\underline{i}$ = current in wire of length $\underline{l}$, in amperes
$\bar{l}$ = directed length of wire, in meters
$\bar{B}$ = external magnetic field at the length of wire, in webers/meter^2.

In particular, for Figure 2, page 9, let

$\bar{F} = \bar{F}_2$ = magnetic force acting on the center of mass of the nearest parallel piece of the wire loop at radial distance $\underline{r} = \underline{s}$ from the center axis of the long wire, in newtons
$i = i_2$ = current induced through the wire loop, in amperes
$\bar{l} = -l\bar{u}_z$ = directed length of the nearest parallel piece of the wire loop, in meters
$\bar{B} = \bar{B}_{1s}$ = external tangential magnetic field due to current $\underline{i}_1$ evaluated at radial distance $\underline{r} = \underline{s}$ from the center axis of the long wire, in webers/meter^2.

By Appendix E, the equation for the radial sinusoidal magnetic force acting on the center of mass of the nearest parallel piece of the wire loop of Figure 2, page 9, is

$$\bar{F}_2 = F_{20} \sin(2wt+2\alpha - (\tan^{-1}(X_1/R_1))) \bar{u}_r + \frac{X_1}{((R_1)^2+(X_1)^2)^{\frac{1}{2}}} F_{20} \bar{u}_r \text{ newtons} \qquad (11)$$

where

$$F_{20} = \frac{N\mu_0{}^2\mu_r{}^2 l^2 I_1{}^2 K_G f}{4\pi s(a+l)((R_1)^2+(X_1)^2)^{\frac{1}{2}}} \text{ newtons}$$

$$K_G = \ln\left|\frac{s+(a \cos\phi)}{s}\right| + \tfrac{1}{2}\ln\left|\frac{s^2+(2sa \cos\phi)+a^2}{s^2+(2sa \cos\phi)+(a^2 \cos^2\phi)}\right|$$

$$w = 2\pi f \text{ hertz}$$

$$X_1 = wL_1 \text{ ohms.}$$

The radial direction of the magnetic force $\bar{F}_2$ in Equation (11) above comes about quite naturally from the vector cross product and the previous definitions for the directions of current $\underline{i}_1$ in the long wire and induced current $\underline{i}_2$ in the wire loop. By the right-hand rule (implicit in the vector cross product of Equation (10), page 21), when $\bar{F}_2$ is instantaneously greater than zero, the magnetic force acting on the center of mass of the nearest parallel piece of the wire loop is radially outward from the long wire (repulsive).

In a similar manner, Equation (10), page 21, can be used to determine the magnetic force acting on the center of mass of the farthest parallel piece of the wire loop. In particular, for Figure 2, page 9, let

$\bar{F} = \bar{F}_2{}'$ = magnetic force acting on the center of mass of the farthest parallel piece of the wire loop at radial distance $\underline{r} = \underline{c}$ from the center axis of the long wire, in newtons

$i = i_2$ = current induced through the wire loop, in amperes

$\bar{l} = l\bar{u}_z$ = directed length of the farthest parallel piece of the wire loop, in meters

$\bar{B} = \bar{B}_{1c}$ = external tangential magnetic field due to current $\underline{i_1}$ evaluated at radial distance $\underline{r} = \underline{c}$ from the center axis of the long wire, in webers/meter^2.

By the law of cosines, the radial distance $\underline{r} = \underline{c}$ from the center axis of the long wire to the center axis of the farthest parallel piece of the wire loop of Figure 2, page 9, is

$$c = (s^2+(2sa\ \cos\phi)+a^2)^{\frac{1}{2}} \text{ meters.}$$

Using a procedure analgous to that of Appendix E, noting that $\bar{l} = l\bar{u}_z$, and evaluating the external tangential magnetic field due to current $\underline{i_1}$ at radial distance $\underline{r} = \underline{c}$ from the center axis of the long wire, the radial sinusoidal magnetic force acting on the center of mass of the farthest parallel piece of the wire loop of Figure 2, page 9, is

$$\bar{F}_2{}' = -F_{20}{}'\ \sin(2wt+2\alpha-(\tan^{-1}(X_1/R_1)))\ \bar{u}_r - \frac{X_1}{((R_1)^2+(X_1)^2)^{\frac{1}{2}}} F_{20}{}'\ \bar{u}_r \text{ newtons} \qquad (12)$$

where

$$F_{20}' = \frac{N\mu_0^2\mu_r^2 l^2 I_1^2 K_G f}{4\pi c(a+l)((R_1)^2+(X_1)^2)^{\frac{1}{2}}} \text{ newtons}$$

$$K_G = \ln\left|\frac{s+(a\cos\phi)}{s}\right| + \tfrac{1}{2}\ln\left|\frac{s^2+(2sa\cos\phi)+a^2}{s^2+(2sa\cos\phi)+(a^2\cos^2\phi)}\right|$$

$$c = (s^2+(2sa\cos\phi)+a^2)^{\frac{1}{2}} \text{ meters}$$

$$w = 2\pi f \text{ hertz}$$

$$X_1 = wL_1 \text{ ohms.}$$

By the right-hand rule (implicit in the vector cross product of Equation (10), page 21), when the magnetic force $\bar{F}_2$ of Equation (11), page 22, is instantaneously greater than zero, the magnetic force $\bar{F}_2'$ of Equation (12), page 23, acting on the center of mass of the farthest parallel piece of the wire loop is instantaneously less than zero and radially inward toward the long wire (attractive).

Where the rms values, $\underline{F}_2$ and $\underline{F}_2'$, of the sinusoidal magnetic forces of Equations (11) and (12) are of interest, the definition of the root-mean-square value of a time-periodic function can be used to show that

$$\underline{F}_2 = \frac{F_{20}}{\sqrt{2}}(1+(2(X_1)^2/((R_1)^2+(X_1)^2)))^{\frac{1}{2}} \text{ newtons} \quad (11a)$$

$$\underline{F}_2' = \frac{F_{20}'}{\sqrt{2}}(1+(2(X_1)^2/((R_1)^2+(X_1)^2)))^{\frac{1}{2}} \text{ newtons} \quad (12a)$$

where $\underline{F}_{20}$, $\underline{F}_{20}'$, and $\underline{X}_1$ are as defined in Equation (11), page 22, and Equation (12), page 23.

Note that a restriction on Equation (11), page 22, and Equation (12), page 23, is that the radius of the "thin" wire in the wire loop must be much less than the radial distance $\underline{s}$ from the center axis of the long wire to the center axis of the nearest parallel piece of the wire loop of Figure 2, page 9.

Equation (11), page 22, and Equation (12), page 23, are useful in determining the magnitude of the resultant magnetic force on the wire loop.

8. MAGNITUDE OF RESULTANT MAGNETIC FORCE

Although Equation (11), page 22, and Equation (12), page 23, describe the sinusoidal magnetic forces $\bar{F}_2$ and $\bar{F}_2'$ acting on the centers of mass of the nearest and farthest parallel pieces of the wire loop respectively, the resultant magnetic force on the wire loop of Figure 2, page 9, is the vector sum of the forces on the four sides of the wire loop.

By the magnetic force equation, Equation (10), page 21, at any instant of time, the sinusoidal magnetic forces acting on the centers of mass of the wires making the parallel sides of width <u>a</u> in the wire loop are equal in magnitude but in opposite directions so that their vector sum is zero.

By Equation (11), page 22, and Equation (12), page 23, the sinusoidal magnetic forces $\bar{F}_2$ and $\bar{F}_2'$, at any instant of time, are unequal in magnitude and acting in radially opposite directions on the centers of mass of the nearest and farthest parallel pieces of length <u>l</u> of the wire loop.

By simple vector addition, then, the magnitude of the resultant magnetic force acting on the center of mass of the entire wire loop of Figure 2, page 9, is

$$|\bar{F}_2+\bar{F}_2'| = (|\bar{F}_2|^2-(2|\bar{F}_2||\bar{F}_2'| \cos\theta)+|\bar{F}_2'|^2)^{\frac{1}{2}} \text{ newtons} \quad (13)$$

where

$$\theta = \tan^{-1}((a \sin\phi)/(s+(a \cos\phi))).$$

By Equation (11), page 22, and Equation (12), page 23, the magnitude of the sinusoidal magnetic force $\bar{F}_2$, $|\bar{F}_2|$, is always greater than the magnitude of the sinusoidal magnetic force $\bar{F}_2'$, $|\bar{F}_2'|$, since radial distance s is always less than radial distance c where $c = (s^2+(2sa \cos\phi)+a^2)^{\frac{1}{2}}$. In particular, when the radial distance c from the center axis of the long wire to the center axis of the farthest parallel piece of the wire loop is much greater than the radial distance s from the center axis of the long wire to the center axis of the nearest parallel piece of the wire loop, then $|\bar{F}_2'|$ is much less than $|\bar{F}_2|$.

By Equation (13) above, when the magnitude of the sinusoidal magnetic force $\bar{F}_2'$ acting on the center of mass of the farthest parallel piece of the wire loop is much less than the magnitude of the sinusoidal magnetic force $\bar{F}_2$ acting on the center of mass of the nearest parallel piece of the wire loop (as when c is much greater than s), then the magnitude of the resultant magnetic force acting on the center of mass of the entire wire loop of Figure 2, page 9, is approximately

$$|\bar{F}_2 + \bar{F}_2'| \approx |\bar{F}_2| \text{ newtons} \qquad (14)$$

where $c = (s^2+(2sa\ \cos\phi)+a^2)^{\frac{1}{2}}$ is much greater than s.

Equation (14) above is a good, simple approximation for the magnitude of the resultant magnetic force on the center of mass of the entire wire loop of Figure 2, page 9, when the radial distance c is much greater than the radial distance s (which implies that the width a of the wire loop is much greater than s).

9. COMPUTER PROGRAM

9.1 PROGRAMMING LANGUAGE

The program magforce.ipli was written in IPLI (a basic form of PL/I) because this language is compatible for use with time-sharing-option (TSO) computer terminals available at Pennsylvania Power and Light Company in Allentown, Pennsylvania.

9.2 DESCRIPTION OF PROGRAM

The program magforce.ipli is based on the equations derived in this paper. Using its own prompting subroutine called mag.ipli , the program explains data entry to the user. The prompting subroutine graphically depicts the special orientation of the wire loop with respect to the long wire as shown in Figure 2, page 9, and describes all required input data (including required dimensional units) in detail. After all required input data has been entered, the program uses an iterative process to calculate the magnitude of the magnetic field effect variables.

9.3 EXPLANATION OF DATA ENTRY

The program prompts the user for the required input data when the user so desires. The program requires the following data in order:

- R_1 = radius of the long wire, in meters
- f = frequency of the sinusoidal current in the long wire, in hertz
- I_1 = rms value of the sinusoidal current in the long wire, in amperes
- s_{min} = minimum distance from the center axis of the long wire to the center axis of the nearest parallel piece of the wire loop, in meters
- s_{max} = maximum distance from the center axis of the long wire to the center axis of the nearest parallel piece of the wire loop, in meters
- s_{inc} = incremental distance from the center axis of the long wire to the center axis of the nearest parallel piece of the wire loop, in meters
- ϕ_d = angle that the plane of the wire loop makes with the plane containing the long wire and the nearest parallel piece of the wire loop, in degrees between zero and ninety
- R_2 = radius of the thin wire in the wire loop, in meters
- N = number of turns in the wire loop
- l = length of the wire loop, in meters
- a = width of the wire loop, in meters
- R_l = resistance per unit length of the thin wire of the wire loop, in ohms/meter
- L_l = inductance per unit length of the thin wire of the wire loop, in henries/meter.

9.4 EXPLANATION OF DATA OUTPUT

As the radial distance $\underline{s}$ from the center axis of the long wire to the center axis of the nearest parallel piece of the wire loop varies from $\underline{s}_{min}$ to $\underline{s}_{max}$ by increments of $\underline{s}_{inc}$, the program <u>magforce.ipli</u> calculates:

B_1 = rms value of the tangential magnetic field at radial distance s from the center axis of the long wire to the center axis of the nearest parallel piece of the wire loop, in webers/meter^2

E_2 = rms value of the electromotive force induced in the wire loop when the ends of the wire loop are infinitesimally open-circuited, in volts

I_2 = rms value of the current induced in the wire loop, in amperes

F_2 = rms value of the radial magnetic force acting on the center of mass of the nearest parallel piece of the wire loop, in newtons.

The program also prints out all of the given input data entered.

9.5 USE OF THE PROGRAM

The program magforce.ipli can be run from any TSO terminal at Pennsylvania Power and Light Company, Two North Ninth Street, Allentown, Pennsylvania. Simply turn the TSO terminal on and type the following:

```
logon distdev/section
run magforce.ipli
```

The program will then respond with a brief description of its purpose as well as its last revision date and will ask the user if he desires an explanation of data entry.

The program is useful for verifying calculations of examples based on the equations derived in this paper.

10. EXAMPLES

The following two examples were selected to show some typical values of the magnitude of the induced voltage and the magnetic force due to a sinusoidal current in a long, thin, straight wire whose current magnitude is typical of that found in electric power distribution lines.

10.1 EXAMPLE 1

Example 1 illustrates the magnitude of the induced voltage due to a sinusoidal current in a long, thin, straight wire whose current magnitude is typical of that found in electric power distribution lines.

A long, thin, straight #4/0 ACSR (aluminum cable steel reinforced) wire in air carries a sinusoidal current $\underline{i}_1 = \sqrt{2}\ I_1 \sin(2\pi 60t)$ amperes. See Figure 3, page 33. Directly beneath the current-carrying #4/0 ACSR wire are two other long, thin, straight #4/0 ACSR wires grounded together at one end; parallel to each other and to the current-carrying #4/0 ACSR wire; and lying in the same vertical plane. The distance from the current-carrying #4/0 ACSR wire to the nearest parallel #4/0 ACSR wire is $\underline{s}$ meters, and the two #4/0 ACSR wires grounded together at one end are $\underline{a}$ meters

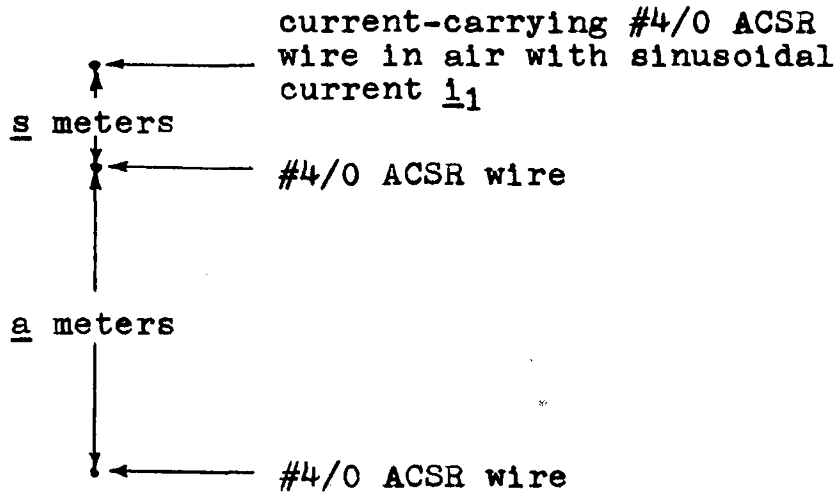


Figure 3. Special orientation of Example 1.

apart.

Comparing the special orientation of Example 1 shown in Figure 3, page 33, with the special orientation and coordinate system of Figure 1, page 4, and Figure 2, page 9:

$$f = 60 \text{ hertz}$$
$$\phi = 0$$
$$N = 1.$$

From tables, the values of the standard constants μ_0 and μ_r are:

$$\mu_0 = 4\pi \times 10^{-7} \text{ webers/(ampere meter)}$$
$$\mu_r = 1.$$

Using Equation (7), page 17, the rms value of the sinusoidal emf per unit length induced between the ungrounded ends of the two #4/0 ACSR wires <u>a</u> meters apart is, for $\phi = 0$,

$$(E_2/l) = (N\mu_0\mu_r I_1 f) \ln(1+(a/s)) \text{ volts/meter.}$$

Now, for example, a typical maximum value of current carried by a #4/0 ACSR wire is $I_1 = 340$ amperes while typical parameters for an electric power distribution line are <u>s</u> = 0.3 meter and <u>a</u> = 0.9 meter.

Substituting this remaining given data,

$$I_1 = 340 \text{ amperes}$$
$$s = 0.3 \text{ meter}$$
$$a = 0.9 \text{ meter,}$$

yields

$$(E_2/l) = 0.0355 \text{ volts/meter.}$$

If the two #4/0 ACSR wires grounded together at one end parallel the current-carrying #4/0 ACSR wire over a length of 4 kilometers, an rms induced voltage of 142 volts appears between their ungrounded ends.

A person coming into contact with the ungrounded ends of the two #4/0 ACSR wires in Example 1 and thinking them to be otherwise unenergized might be surprised to experience a voltage whose magnitude is greater than that found between the terminals of a typical household electrical outlet.

Figure 4, page 36, shows calculated rms values of induced emf per unit length, (E_2/l), obtained by using the computer program magforce.ipli for the special orientation of Figure 3, page 33, for

$$f = 60 \text{ hertz}$$
$$\phi = 0$$
$$N = 1$$

and

$$a = 0.9 \text{ meter}$$

where

I_1 = 340 amperes, 255 amperes, or 170 amperes

and s varies between

$$s_{min} = 0.3 \text{ meter}$$
$$s_{max} = 1.0 \text{ meter.}$$

Figure 5, page 37, shows calculated rms values of induced emf per unit length, (E_2/l), obtained by using

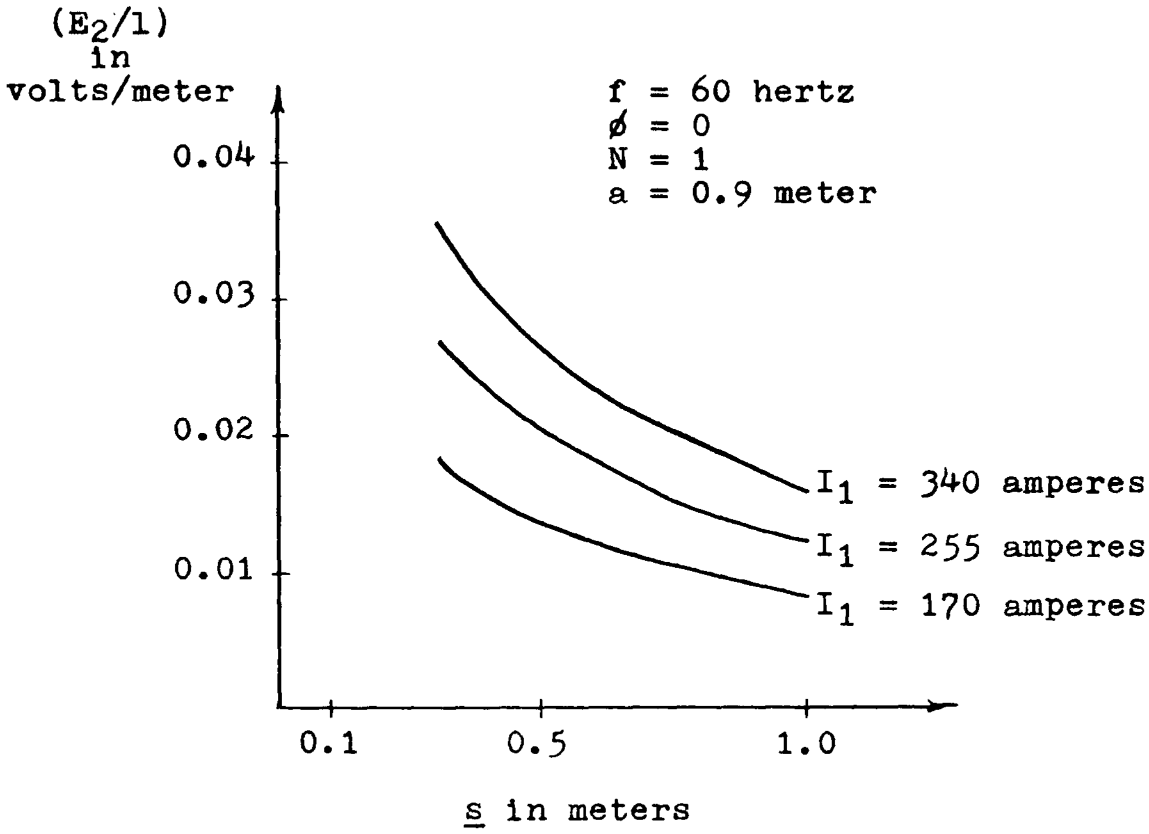


Figure 4. Calculated rms values of induced emf per unit length, (E_2/l), for $\underline{a}$ = 0.9 meter in Example 1 as $\underline{s}$ varies for selected values of $\underline{I}_1$.

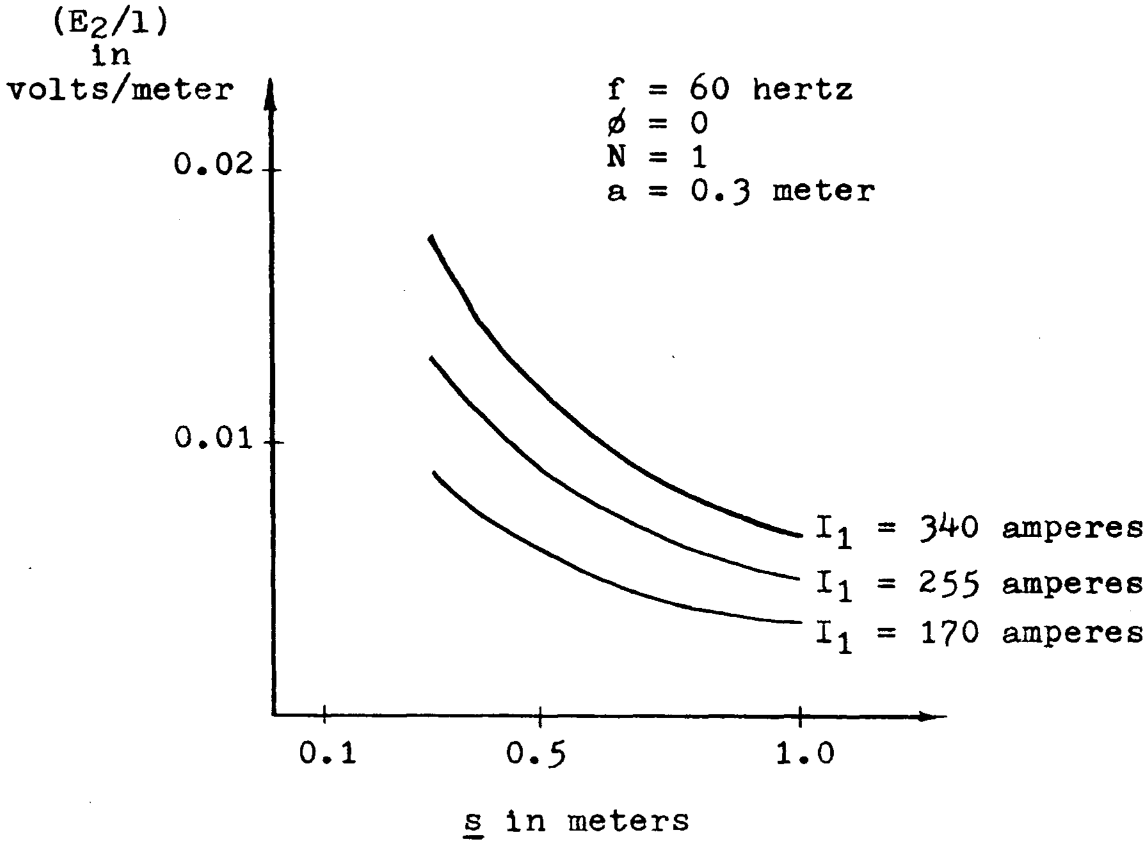


Figure 5. Calculated rms values of induced emf per unit length, (E_2/l), for $\underline{a}$ = 0.3 meter in Example 1 as $\underline{s}$ varies for selected values of $\underline{I}_1$.

the computer program magforce.ipli for the special orientation of Figure 3, page 33, for

$$f = 60 \text{ hertz}$$
$$\phi = 0$$
$$N = 1$$

and

$$a = 0.3 \text{ meter}$$

where

I_1 = 340 amperes, 255 amperes, or 170 amperes

and s varies between

$$s_{min} = 0.3 \text{ meter}$$
$$s_{max} = 1.0 \text{ meter.}$$

10.2 EXAMPLE 2

Example 2 illustrates the magnitude of the magnetic force due to a sinusoidal current in a long, thin, straight wire whose current magnitude is typical of that found in electric power distribution lines.

A long, thin, straight #4/0 ACSR wire in air carries a sinusoidal current $\underline{i}_1 = \sqrt{2}\, I_1 \sin(2\pi 60t)$ amperes. A nearby "antenna" consists of a rectangular wire loop of #6 copper wire specially oriented with respect to the current-carrying #4/0 ACSR wire as shown in Figure 2, page 9. The inductive reactance per unit length of the #6 copper wire is much less than its resistance per unit length so that the inductance per unit length relative to the resistance per unit length of the wire loop is negligible.

Given data for this example is:

$$
\begin{aligned}
R_1 &= 0.0072 \text{ meter} \\
f &= 60 \text{ hertz} \\
I_1 &= 340 \text{ amperes} \\
s &= 0.5 \text{ meter} \\
\phi &= 0 \\
R_2 &= 0.0021 \text{ meter} \\
N &= 1 \\
l &= 6 \text{ meters} \\
a &= 6 \text{ meters} \\
R_l &= 0.00136 \text{ ohms/meter} \\
L_l &= 0.
\end{aligned}
$$

The values of the standard constants μ_0 and μ_r are as given on page 34 of Example 1.

From the data given above for Example 2, the radial distance $\underline{c} = \underline{s} + \underline{a} = 6.5$ meters from the center axis of the long wire to the center axis of the farthest parallel piece of the wire loop is much greater than the radial distance $\underline{s} = 0.5$ meter from the center axis of the long wire to the center axis of the nearest parallel piece of the wire loop. Using Equation (14), page 28, and Equation (11a), page 24, the maximum rms value of the magnitude of the resultant magnetic force acting on the center of mass of the entire antenna is, for $\phi = 0$ and $L_l = 0$,

$$F_2 = \frac{1}{\sqrt{2}} \frac{N\mu_0{}^2\mu_r{}^2 l^2 I_1{}^2 f}{4\pi s(a+l)R_l} \ln(1+(a/s)) \text{ newtons}$$

and substituting the remaining given data,

$$F_2 = 0.00697 \text{ newtons.}$$

It is readily apparent that this maximum worst case rms value, F_2, of the magnitude of the resultant magnetic force acting on the center of mass of the entire antenna will have a negligible effect on any motion of the antenna. (F_2 is 0.00157 pounds where 1 newton equals 0.2248 pounds.) Furthermore, as explained previously, when $\bar{F}_2$ is instantaneously greater than zero, the magnetic force acting on the center of mass of the entire antenna (wire loop) of Example 2 will be radially outward from the long wire (repulsive).

Figure 6, page 41, shows calculated values of magnetic force F_2 obtained by using the computer program magforce.ipli for the special orientation of Figure 2, page 9, for

$$\begin{aligned} R_1 &= 0.0072 \text{ meter} \\ f &= 60 \text{ hertz} \\ \phi &= 0 \\ R_2 &= 0.0021 \text{ meter} \\ N &= 1 \\ l &= 6 \text{ meters} \\ a &= 6 \text{ meters} \\ R_l &= 0.00136 \text{ ohms/meter} \\ L_l &= 0 \end{aligned}$$

where

I_1 = 340 amperes, 255 amperes, or 170 amperes

and $\underline{s}$ varies between

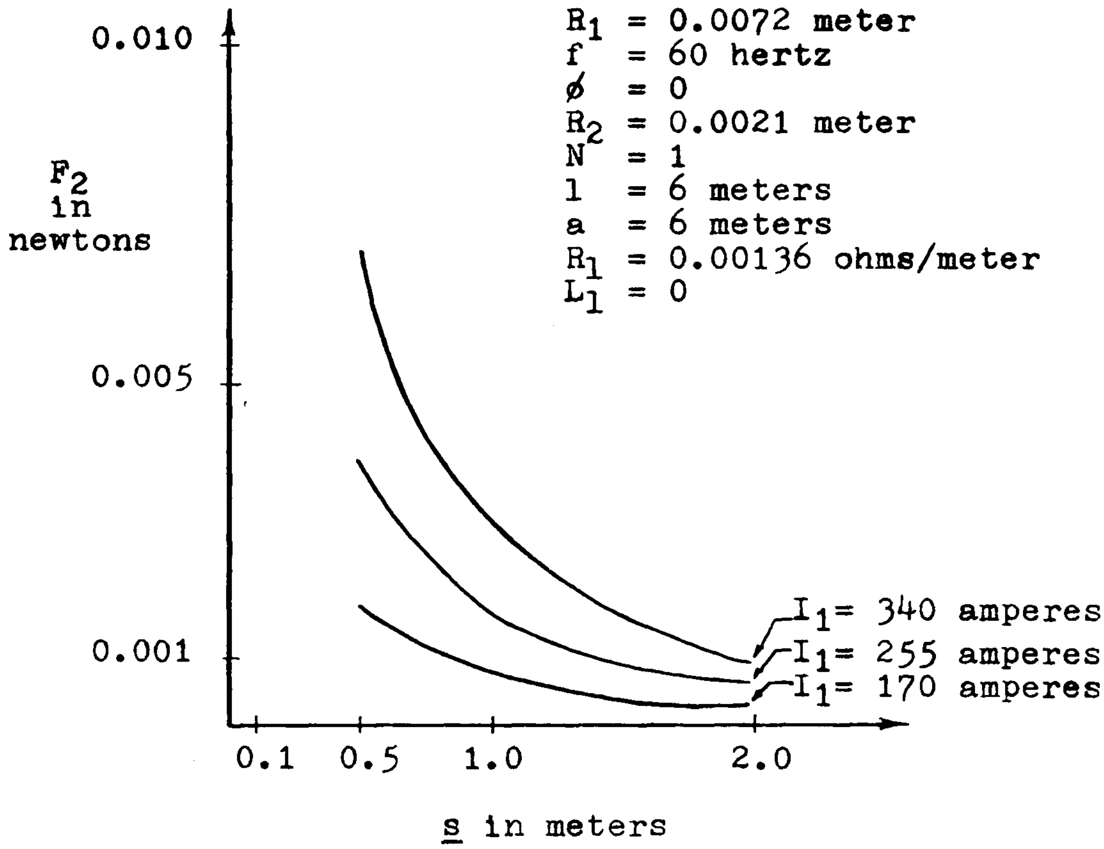


Figure 6. Calculated values of magnetic force F_2 for $\underline{l} = \underline{a} = 6$ meters in Example 2 as $\underline{s}$ varies for selected values of $\underline{I}_1$.

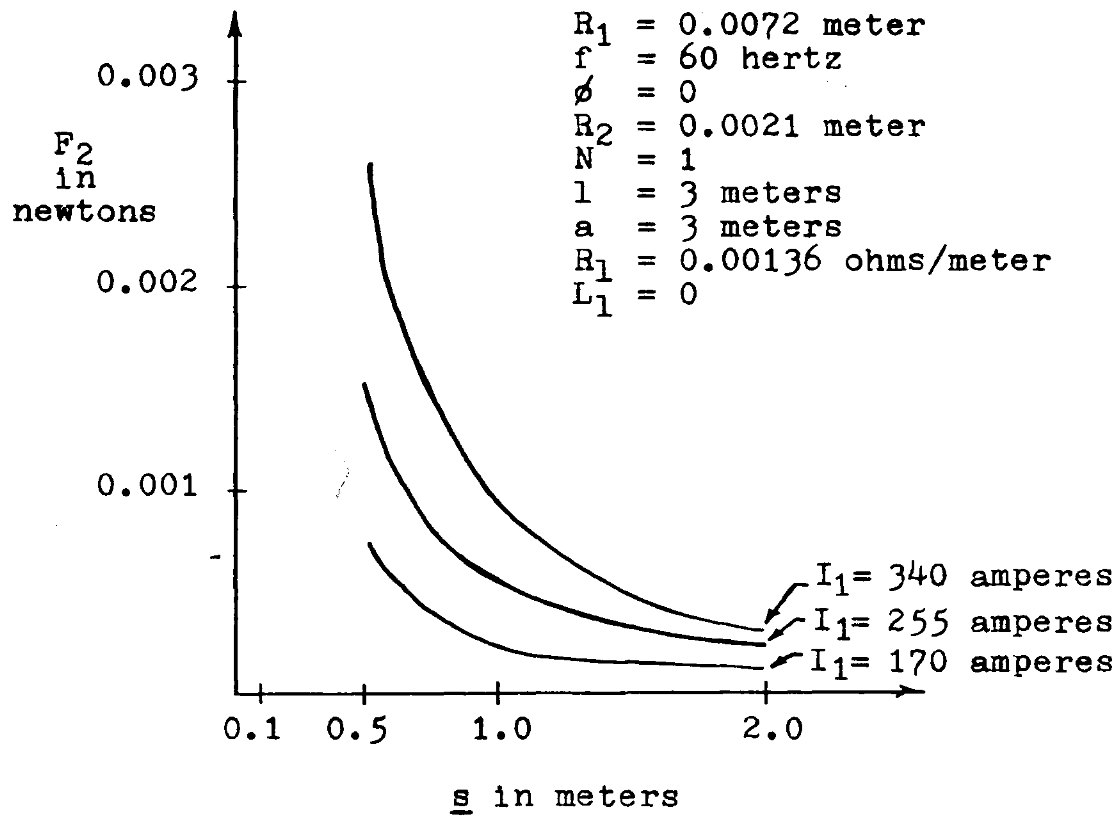


Figure 7. Calculated values of magnetic force F_2 for $\underline{l} = \underline{a} = 3$ meters in Example 2 as $\underline{s}$ varies for selected values of $\underline{I}_1$.

$$s_{min} = 0.5 \text{ meter}$$
$$s_{max} = 2.0 \text{ meters.}$$

Figure 7, page 42, shows calculated values of magnetic force F_2 obtained by using the computer program <u>magforce.ipli</u> for the special orientation of Figure 2, page 9, for

$$R_1 = 0.0072 \text{ meter}$$
$$f = 60 \text{ hertz}$$
$$\phi = 0$$
$$R_2 = 0.0021 \text{ meter}$$
$$N = 1$$
$$l = 3 \text{ meters}$$
$$a = 3 \text{ meters}$$
$$R_l = 0.00136 \text{ ohms/meter}$$
$$L_l = 0$$

where

I_1 = 340 amperes, 255 amperes, or 170 amperes

and <u>s</u> varies between

$$s_{min} = 0.5 \text{ meter}$$
$$s_{max} = 2.0 \text{ meters.}$$

11. CONCLUSION

This paper has thoroughly analyzed in a quantitative manner the magnitude of magnetic field effects due to a steady-state sinusoidally time-varying current in a long, thin, straight wire.

A wire loop specially oriented with respect to the long, thin, straight current-carrying wire was used to determine the magnitude of these magnetic field effects which included: (1) the magnitude of the magnetic field around the long, thin, straight current-carrying wire; (2) the magnitude of the induced voltage in the wire loop specially oriented with respect to the long, thin, straight current-carrying wire; (3) the magnitude of the induced current in the wire loop specially oriented with respect to the long, thin, straight current-carrying wire; and (4) the magnitude of the magnetic force acting on the nearest parallel piece of the wire loop.

It was seen that induced voltages on otherwise isolated or unenergized electric power distribution lines can be on the order of 150 volts, presenting a shock safety hazard comparable to that a person might receive by touching the two terminals of a typical household electrical outlet.

It was also seen that the magnitude of the resultant magnetic force on a wire loop near a long, thin, straight wire carrying a current whose magnitude is typical of electric power distribution lines was on the order of 0.01 newtons and would probably have little effect on any motion of the wire loop.

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APPENDIX A - COORDINATE SYSTEM NOTATION

Refer to Figure 2, page 9. Let $\bar{u}_x$, $\bar{u}_y$, and $\bar{u}_z$ be unit vectors in the $\underline{x}$, $\underline{y}$, and $\underline{z}$ directions respectively of a right-handed rectangular cartesian coordinate system. Let $\bar{u}_x$, $\bar{u}_y$, and $\bar{u}_z$ be represented by the ordered vector triplets (1,0,0), (0,1,0), and (0,0,1) respectively.

The point P_1:(x,y,z) in the right-handed rectangular cartesian coordinate system is equivalent to the point P_1:(r,θ,z) in the right-handed circular cylindrical coordinate system.

Let $\bar{\rho}$ be the distance vector to the point P_1, and let $\bar{\rho}$ be represented by the ordered vector triplet (x,y,z). Let $\bar{r}$ be the vector projection of $\bar{\rho}$ in the z = 0 plane (that is, the distance vector to the point P_0), and let $\bar{r}$ be represented by the ordered vector triplet (x,y,0). Let θ be the angle measured (counterclockwise as seen from the positive $\underline{z}$-axis) from the positive $\underline{x}$-axis to the vector $\bar{r}$. If the magnitude of a vector $\bar{v} = (v_1,v_2,v_3)$ is written as $|v| = v$, then

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

and

$$\begin{aligned}\bar{r} &= x\,\bar{u}_x + y\,\bar{u}_y + 0\,\bar{u}_z \\ &= r\cos\theta\,\bar{u}_x + r\sin\theta\,\bar{u}_y + 0\,\bar{u}_z \\ &= r(\cos\theta\,\bar{u}_x + \sin\theta\,\bar{u}_y + 0\,\bar{u}_z) \\ &= r(\cos\theta,\sin\theta,0)\end{aligned}$$

and

$$\begin{aligned}\bar{\rho} &= x\,\bar{u}_x + y\,\bar{u}_y + z\,\bar{u}_z \\ &= r\cos\theta\,\bar{u}_x + r\sin\theta\,\bar{u}_y + z\,\bar{u}_z \\ &= (r\cos\theta, r\sin\theta, z).\end{aligned}$$

The tangent vectors to curves in the r, θ, and z directions are given by $\partial\bar{\rho}/\partial r$, $\partial\bar{\rho}/\partial\theta$, and $\partial\bar{\rho}/\partial z$ respectively:

$$\partial\bar{\rho}/\partial r = \cos\theta\,\bar{u}_x + \sin\theta\,\bar{u}_y = (\cos\theta,\sin\theta,0)$$

$$\partial\bar{\rho}/\partial\theta = -r\sin\theta\,\bar{u}_x + r\cos\theta\,\bar{u}_y = r(-\sin\theta,\cos\theta,0)$$

$$\partial\bar{\rho}/\partial z = \bar{u}_z = (0,0,1).$$

Thus, the unit vectors tangent to curves in the r, θ, and z directions are given by $\bar{u}_r$, $\bar{u}_\theta$, and $\bar{u}_z$ respectively:

$$\bar{u}_r = \frac{\partial\bar{\rho}/\partial r}{|\partial\bar{\rho}/\partial r|} = \frac{\cos\theta\,\bar{u}_x + \sin\theta\,\bar{u}_y}{(\cos^2\theta + \sin^2\theta)^{\frac{1}{2}}} = (\cos\theta,\sin\theta,0)$$

$$\bar{u}_\theta = \frac{\partial\bar{\rho}/\partial\theta}{|\partial\bar{\rho}/\partial\theta|} = \frac{r(-\sin\theta,\cos\theta,0)}{r} = (-\sin\theta,\cos\theta,0)$$

$$\bar{u}_z = \frac{\partial\bar{\rho}/\partial z}{|\partial\bar{\rho}/\partial z|} = \bar{u}_z = (0,0,1).$$

The coordinate system notation described above in this Appendix A has been adopted and used consistently throughout this paper. Vector quantities are represented in terms of unit vectors $\bar{u}_x$, $\bar{u}_y$, and $\bar{u}_z$ of a right-handed rectangular cartesian coordinate system or in terms of unit vectors $\bar{u}_r$, $\bar{u}_\theta$, and $\bar{u}_z$ of a right-handed circular cylindrical coordinate system, whichever is more appropriate.

APPENDIX B - DETAILS OF FINDING MAGNETIC FLUX

From Equation (5), page 15, the magnetic flux Φ_B due to a magnetic field $\bar{B}$ passing through a surface area $\bar{S}$ consisting of elementary infinitesimal surface areas $d\bar{S}$ is

$$\Phi_B = \int \bar{B} \cdot d\bar{S} \text{ webers.} \tag{5}$$

In particular, for Figure 2, page 9,

$\Phi_B = \Phi_{B_1}$ = magnetic flux passing through the wire loop due to magnetic field $\bar{B}_1$, in webers

$\bar{B} = \bar{B}_1$ = tangential magnetic field due to current $\underline{i}_1$, in webers/meter^2.

By Equation (2), page 12, the tangential magnetic field due to current $\underline{i}_1$ is

$$\bar{B}_1 = \sqrt{2}\,\frac{\mu_0 \mu_r I_1}{2\pi r} \sin(2\pi f t + \alpha)\, \bar{u}_\theta \text{ webers/meter}^2. \tag{2}$$

Let

$$K_B = \sqrt{2}\,\frac{\mu_0 \mu_r I_1}{2\pi} \sin(2\pi f t + \alpha) \text{ webers/meter.}$$

Then

$$\bar{B}_1 = (K_B/r)\, \bar{u}_\theta \text{ webers/meter .}$$

Or, using the notation of Appendix A,

$$\bar{B}_1 = (K_B/r)(-\sin\theta, \cos\theta, 0) \text{ webers/meter}^2.$$

Refer to Figure 8, page 51. Variables for Figure 8 are as defined for Figure 1, page 4, and Figure 2,

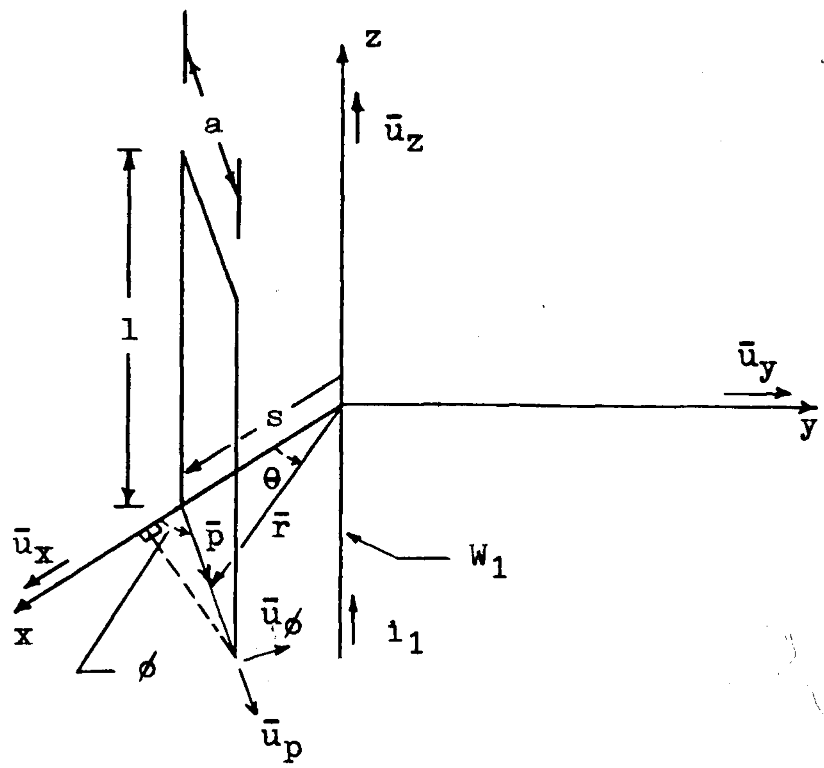

Figure 8. Special orientation for finding magnetic flux.

page 9. In addition, using the notation of Appendix A, ϕ is a constant angle measured (counterclockwise as seen from the positive z-axis) from the positive x-axis to the vector $\bar{p}$. The angle ϕ is a constant in the closed interval zero to $(\pi/2)$, meaning that ϕ may be zero or $(\pi/2)$ or an angle between zero and $(\pi/2)$. $\bar{p}$ is a distance vector whose direction is from the positive x-axis along the width of the wire loop in the plane $z = 0$. $\bar{u}_\phi$ and $\bar{u}_p$ are unit vectors in the ϕ and $\bar{p}$ directions respectively. Then

$$\begin{aligned} \bar{p} &= p(\cos\phi, \sin\phi, 0) \\ \bar{u}_p &= (\cos\phi, \sin\phi, 0) \\ \bar{u}_\phi &= (-\sin\phi, \cos\phi, 0) \end{aligned}$$

where

$$\begin{aligned} \tan\phi &= y/(x-s) \\ \sin\phi &= y/p \\ \cos\phi &= (x-s)/p \end{aligned}$$

$$\begin{aligned} \tan\theta &= y/x = (p \sin\phi)/(s+(p \cos\phi)) \\ \sin\theta &= y/r = (p \sin\phi)/r \\ \cos\theta &= x/r = (s+(p \cos\phi))/r. \end{aligned}$$

Then using the equation above for $\tan\theta$,

$$p = \frac{-s \tan\theta}{\cos\phi \tan\theta - \sin\phi} \text{ meters}$$

$$dp = \frac{s \sin\phi \sec^2\theta \, d\theta}{(\cos\phi \tan\theta - \sin\phi)^2} \text{ meters}$$

and using the equations above for $\cos\theta$ and p,

$$\frac{1}{r} = \frac{\cos\theta(\cos\phi \tan\theta - \sin\phi)}{-s \sin\phi} \text{ meters}^{-1}.$$

Let the surface area under consideration in Equation (5) be the rectangular area of the wire loop of Figure 8, page 51. Let the direction of the infinitesimal surface areas $d\bar{S}$ be the outward-drawn normal to the rectangular area of the wire loop taken in the $\bar{u}_\phi$ direction. Then

$$d\bar{S} = 1\ dp\ \bar{u}_\phi \text{ meter}^2$$

and

$$\bar{B}_1 \cdot d\bar{S} = (K_B/r)(-\sin\theta, \cos\theta, 0)\ 1\ dp\ \bar{u}_\phi \text{ webers.}$$

By substituting in the equation for $\bar{B}_1 \cdot d\bar{S}$ above the previous expressions for $(1/r)$, dp, and $\bar{u}_\phi$, the equation for $\bar{B}_1 \cdot d\bar{S}$ becomes (after considerable effort)

$$\bar{B}_1 \cdot d\bar{S} = -K_B 1(\text{ctn}\phi \sec^2\theta + \tan\theta(\text{ctn}^2\phi - 1) - 2\text{ctn}\phi)d\theta \text{ webers}$$

where

$$\text{ctn}\phi = \text{cotangent of } \phi.$$

Integrating the value of $\bar{B}_1 \cdot d\bar{S}$ shown above over the rectangular area of the wire loop, the length $\underline{l}$ of the wire loop remains constant while the width of the wire loop varies along $\bar{p}$ from $\underline{p} = 0$ to $\underline{p} = \underline{a}$. As the width of the wire loop varies from zero to $\underline{a}$, the angle $\underline{\theta}$ varies from $\underline{\theta} = 0$ to $\underline{\theta} = \theta_a = \tan^{-1}((a \sin\phi)/(s+(a \sin\phi)))$. Then

$$\int \bar{B}_1 \cdot d\bar{S} = -K_B 1(\text{ctn}\phi \text{INT}_1 + (\text{ctn}^2\phi - 1)\text{INT}_2 - 2\text{ctn}\phi \text{INT}_3) \text{ webers}$$

where

$$INT_1 = \int_0^{\theta_a} \sec^2\theta (\mathrm{ctn}\phi \tan\theta - 1)^{-2}\, d\theta$$

$$INT_2 = \int_0^{\theta_a} \tan\theta (\mathrm{ctn}\phi \tan\theta - 1)^{-2}\, d\theta$$

$$INT_3 = \int_0^{\theta_a} (\mathrm{ctn}\phi \tan\theta - 1)^{-2}\, d\theta$$

$$\theta_a = \tan^{-1}((a \sin\phi)/(s+(a \sin\phi)))$$

$\mathrm{ctn}\phi$ = cotangent of ϕ.

Finally, substituting the results of Appendix C for the values of the three resulting integrals INT_1, INT_2, and INT_3, into the equation for the integral of $\bar{B}_1 \cdot d\bar{S}$ yields (again after considerable effort)

$$\Phi_{B_1} = \int \bar{B}_1 \cdot d\bar{S} = \sqrt{2}\, \frac{\mu_0 \mu_r l I_1 K_G}{2\pi} \sin(2\pi f t + \alpha) \text{ webers} \quad (6)$$

where

$$K_G = \ln\left|\frac{s+(a \cos\phi)}{s}\right| + \tfrac{1}{2}\ln\left|\frac{s^2+(2sa \cos\phi)+a^2}{s^2+(2sa \cos\phi)+(a^2 \cos^2\phi)}\right|$$

ln = natural logarithm to the base *e*.

Equation (6) above is the equation which was to be found for the sinusoidal magnetic flux through the wire loop due to the sinusoidal magnetic field $\bar{B}_1$ around the long wire of Figure 2, page 9.

Note that for given values of $\underline{s}$, $\underline{a}$, and ϕ, $\underline{K}_G$ in Equation (6) is a geometric constant describing the special orientation of the wire loop with respect to the long wire. Writing the geometric constant $\underline{K}_G$ in terms of (a/s), it is easily shown by considering values of ϕ in the closed interval zero to $(\pi/2)$ inclusive that $\underline{K}_G$ as a function of ϕ is maximal for a given value of (a/s) if the angle $\phi = 0$. This property of the geometric constant $\underline{K}_G$ confirms the intuitive idea that for a given value of (a/s), the sinusoidal magnetic flux through the wire loop should be maximal when the angle $\phi = 0$.

APPENDIX C - EVALUATION OF 3 RESULTING INTEGRALS

The three integrals that resulted from finding the magnetic flux Φ_{B_1} as the integral of $\bar{B}_1 \cdot d\bar{S}$ on page 54 of Appendix B are

$$INT_1 = \int_0^{\theta_a} \sec^2\theta(\text{ctn}\phi \tan\theta - 1)^{-2}\, d\theta$$

$$INT_2 = \int_0^{\theta_a} \tan\theta(\text{ctn}\phi \tan\theta - 1)^{-2}\, d\theta$$

$$INT_3 = \int_0^{\theta_a} (\text{ctn}\phi \tan\theta - 1)^{-2}\, d\theta$$

where

$\theta_a \quad = \tan^{-1}((a \sin\phi)/(s+(a \sin\phi)))$

$\text{ctn}\phi$ = cotangent of ϕ.

By substituting to change the variable of integration from $\underline{d\theta}$ to $\underline{dX}$ and the limits of integration from $\underline{\theta_a}$ to $\underline{X_a}$ in the above three integrals, INT_1, INT_2, and INT_3 can be evaluated. Let

$$X = \tan\theta.$$

Then

$$dX = \sec^2\theta\, d\theta$$

and the values of $\underline{X}$ evaluated at $\theta = 0$ and $\theta = \theta_a$ respectively are

$$(X)_{\theta=0} = 0$$

$$(X)_{\theta=\theta_a} = (a \sin\phi)/(s+(a \sin\phi)).$$

Define

$$X_a = (a \sin\phi)/(s+(a \sin\phi)).$$

Then INT_1 evaluates easily as

$$INT_1 = \int_0^{X_a} (\text{ctn}\phi\ X - 1)^{-2}\ dX = (a \sin\phi)/s.$$

Similarly, the expressions for INT_2 and INT_3 become

$$INT_2 = \int_0^{X_a} X(\text{ctn}\phi\ X - 1)^{-2}(1+X^2)\ dX$$

$$INT_3 = \int_0^{X_a} (\text{ctn}\phi\ X - 1)^{-2}(1+X^2)\ dX.$$

The integrals INT_2 and INT_3 above, however, do not evaluate as easily as the integral INT_1.

The integrands of both INT_2 and INT_3 are proper rational fractions in which at least one of the irreducible real factors of the denominator is quadratic. By the method of partial fractions, the rational expression of each integrand can be decomposed into a sum of partial fractions, and each partial fraction can then be integrated separately.

The decomposition of the rational expression of each integrand into a sum of partial fractions is a lengthy process and results in four separate integrals for INT_2 and four separate integrals for INT_3. Each of these eight integrals is easily evaluated, however. After considerable effort,

$$\begin{aligned} INT_2 = {} & ((1-b^2)/(b^2+1)^2)(\ln|s/(s+(a\cos\phi))|) \\ & + (b^2+1)^{-1}((a\cos\phi)/s) \\ & + ((1-b^2)/(b^2+1)^2)\left(\tfrac{1}{2}\ln\left|\frac{s^2+(2sa\cos\phi)+a^2}{s^2+(2sa\cos\phi)+(a^2\cos^2\phi)}\right|\right) \\ & + ((-2b)/(b^2+1)^2)(\tan^{-1}((a\sin\phi)/(s+(a\sin\phi)))) \end{aligned}$$

$$\begin{aligned} INT_3 = {} & ((-2b)/(b^2+1)^2)(\ln|s/(s+(a\cos\phi))|) \\ & + (b/(b^2+1))((a\cos\phi)/s) \\ & + (b/(b^2+1)^2)\left(\ln\left|\frac{s^2+(2sa\cos\phi)+a^2}{s^2+(2sa\cos\phi)+(a^2\cos^2\phi)}\right|\right) \\ & - ((b^2-1)/(b^2+1)^2)(\tan^{-1}((a\sin\phi)/(s+(a\sin\phi)))) \end{aligned}$$

where

b = ctnϕ = cotangent of ϕ

ln = natural logarithm to the base $\underline{e}$.

The three integrals INT_1, INT_2, and INT_3 evaluated in this Appendix C are those required for determining the magnetic flux Φ_{B_1} as the integral of $\bar{B}_1 \cdot d\bar{S}$ on page 54 of Appendix B.

APPENDIX D - DETAILS OF APPLYING OHM'S LAW

From Ohm's Law, Equation (8), page 19, the current $\underline{i}$ through an impedance $\underline{Z}$ with voltage $\underline{v}$ across the impedance is

$$i = \frac{v}{Z} \text{ amperes.} \qquad (8)$$

In particular, for Figure 2, page 9,

$i = i_2$ = current induced through the wire loop, in amperes
$v = e_2$ = emf induced between infinitesimally-opened ends of the wire loop, in volts
$Z = Z_2$ = impedance of the wire loop, in ohms.

By Equation (7), page 17, the sinusoidal emf induced in the wire loop is

$$e_2 = \sqrt{2}\,(-N\mu_0\mu_r l I_1 K_G f)\,\sin(2\pi ft+\alpha+(\pi/2)) \text{ volts} \qquad (7)$$

where

$$K_G = \ln\left|\frac{s+(a\cos\phi)}{s}\right| + \tfrac{1}{2}\ln\left|\frac{s^2+(2sa\cos\phi)+a^2}{s^2+(2sa\cos\phi)+(a^2\cos^2\phi)}\right|.$$

From Figure 1, page 4, the impedance of the wire loop is

$$Z_2 = 2(a+l)(R_1+j2\pi fL_1) \text{ ohms.}$$

Using phasor representation for the emf $\underline{e}_2$ induced in the wire loop and writing complex impedance $\underline{Z}_2$ in polar form, Equation (8) above yields

$$i_2 = \sqrt{2}\ I_2 \sin(wt+\alpha+(\pi/2)-(\tan^{-1}(X_1/R_1))) \text{ amperes} \quad (9)$$

where

$$I_2 = \frac{(-N\mu_0\mu_r l I_1 K_G f)}{2(a+l)((R_1)^2+(X_1)^2)^{\frac{1}{2}}} \text{ amperes}$$

$$K_G = \ln\left|\frac{s+(a\cos\phi)}{s}\right| + \tfrac{1}{2}\ln\left|\frac{s^2+(2sa\cos\phi)+a^2}{s^2+(2sa\cos\phi)+(a^2\cos^2\phi)}\right|$$

$$w = 2\pi f \quad \text{hertz}$$

$$X_1 = wL_1 \quad \text{ohms.}$$

Equation (9) above is the equation which was to be found for the sinusoidal current induced in the wire loop of Figure 2, page 9.

APPENDIX E - DETAILS OF MAGNETIC FORCE EQUATION

From the magnetic force equation, Equation (10), page 21, the magnetic force $\bar{F}$ acting on the center of mass of a wire of length $\underline{l}$ carrying a current $\underline{i}$ in an external magnetic field $\bar{B}$ is

$$\bar{F} = i\bar{l} \times \bar{B} \text{ newtons.} \qquad (10)$$

In particular, for Figure 2, page 9, let

$\bar{F} = \bar{F}_2$ = magnetic force acting on the center of mass of the nearest parallel piece of the wire loop at radial distance $\underline{r} = \underline{s}$ from the center axis of the long wire, in newtons

$i = i_2$ = current induced through the wire loop, in amperes

$\bar{l} = -l\bar{u}_z$ = directed length of the nearest parallel piece of the wire loop, in meters

$\bar{B} = \bar{B}_{1s}$ = external tangential magnetic field due to current $\underline{i_1}$ evaluated at radial distance $\underline{r} = \underline{s}$ from the center axis of the long wire, in webers/meter^2.

By Equation (2), page 12, the external tangential magnetic field due to current $\underline{i_1}$ and evaluated at radial distance $\underline{r} = \underline{s}$ from the center axis of the long wire is

$$\bar{B}_{1s} = \sqrt{2}\,\frac{\mu_0\mu_r I_1}{2\pi s}\sin(2\pi ft+\alpha)\,\bar{u}_\theta \text{ webers/meter}^2. \qquad (2)$$

By Equation (9), page 19, the sinusoidal current induced in the wire loop of Figure 2, page 9, is

$$i_2 = \sqrt{2}\ I_2\ \sin(wt+\alpha+(\pi/2)-(\tan^{-1}(X_1/R_1)))\ \text{amperes} \quad (9)$$

where

$$I_2 = \frac{(-N\mu_0\mu_r lI_1K_Gf)}{2(a+1)((R_1)^2+(X_1)^2)^{\frac{1}{2}}}\ \text{amperes}$$

$$K_G = \ln\left|\frac{s+(a\ \cos\phi)}{s}\right| + \tfrac{1}{2}\ln\left|\frac{s^2+(2sa\ \cos\phi)+a^2}{s^2+(2sa\ \cos\phi)+(a^2\ \cos^2\phi)}\right|$$

$$w = 2\pi f\ \text{hertz}$$

$$X_1 = wL_1\ \text{ohms}$$

and

$$\bar{l} = -l\ \bar{u}_z\ \text{meters.}$$

By substituting the expressions above for $\bar{B}_{1s}$, $\underline{i}_2$, and $\bar{l}$, into Equation (10) and using the trigonometric identities $\sin(D+(\pi/2)) = \cos(D)$ and $\sin(C)\cos(D) = \frac{1}{2}(\sin(C+D) + \sin(C-D))$, the equation for the radial sinusoidal magnetic force acting on the center of mass of the nearest parallel piece of the wire loop of Figure 2, page 9, becomes (after some effort)

$$\bar{F}_2 = F_{20}\sin(2wt+2\alpha-(\tan^{-1}(X_1/R_1)))\ \bar{u}_r + \frac{X_1}{((R_1)^2+(X_1)^2)^{\frac{1}{2}}}\ F_{20}\ \bar{u}_r\ \text{newtons} \quad (11)$$

where

$$F_{20} = \frac{N\mu_0{}^2\mu_r{}^2 l^2 I_1{}^2 K_G f}{4\pi s(a+1)((R_1)^2+(X_1)^2)^{\frac{1}{2}}} \quad \text{newtons}$$

$$K_G = \ln\left|\frac{s+(a\cos\phi)}{s}\right| + \tfrac{1}{2}\ln\left|\frac{s^2+(2sa\cos\phi)+a^2}{s^2+(2sa\cos\phi)+(a^2\cos^2\phi)}\right|$$

$$w = 2\pi f \quad \text{hertz}$$

$$X_1 = wL_1 \quad \text{ohms.}$$

The radial direction of the magnetic force $\bar{F}_2$ in Equation (11) comes about quite naturally from the vector cross product and the previous definitions for the directions of current $\underline{i}_1$ in the long wire and induced current $\underline{i}_2$ in the wire loop. By the right-hand rule (implicit in the vector cross product of Equation (10)), when $\bar{F}_2$ is instantaneously greater than zero, the magnetic force acting on the center of mass of the nearest parallel piece of the wire loop is radially outward from the long wire (repulsive).

VITA

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